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ON THE COMMUTATIVITY AND EQUALITY OF TWO PROJECTIONS

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ABSTRACT

A simple and a much shorter proof of Rehder's theorem is presented and an attempt is made to find the condition under which two projectios are equal.

INTRODUCTION

Let H be a Hilbert space and B(H) denote the algebra of bounded linear operators on H. A projection in B(H) can be defined as an operator P which is self-adjoint and idempotent. We say that two projections P and Q are orthogonal if PQ = 0. In [1] Rehder has shown that for any two projections P,Q in B(H), the commutativity relation PQ = QP is equivalent to PQP = QPQ. A Rehder's proof is spread over three pages and seems to be a complicated one. In this note we aim at giving a straight forward and much shorter proof of the result. Motivated by this result we shall also make an attempt to find the condition under which two projections could be equal.

PROPOSITION I.

If P and Q are any two projections in B(H) such that PQP = QPQ then P and Q commute.

PROOF

We have

$$(I - P)PQP(I - P) = (I - P)QPQ(I - P) = (I - P)QPPQ(I - P)$$
$$= [QP(I - P)]^*[PQ(I - P)]$$

But

$$(I - P)PQP(I - P) = 0$$

Therefore